10.7 Ratio and Root Tests

given $\sum_{k=1}^{\infty} a_k$, if $\sum_{k=1}^{\infty} |a_k|$ converges, then series $\sum_{k=1}^{\infty} a_k$ converges absolutely

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

so, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely

if a series converges absolutely, then it converges

if $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ does not, then $\sum_{k=1}^{\infty} a_k$ converges conditionally

e.g. $\sum_{k=1}^{\infty} \frac{(-1)}{k}$ converges but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

Ratio and Root Tests tell us if a series converges absolutely
(and therefore converges)

Ratio Test

given $\sum_{k=1}^{\infty} a_k$, the series converges absolutely (and therefore converges)

if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$

series diverges if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| > 1$

test is inconclusive if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = 1$

why? if $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = r$, then this says if k is large,

$|a_{k+1}| \approx r\,|a_k|$ → like a geometric series which converges if $|r| < 1$

this is why we need $\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$

<u>example</u> $\sum_{k=1}^{\infty} \underbrace{k\left(\frac{1}{4}\right)^k}_{a_k}$

$a_k = k\left(\frac{1}{4}\right)^k \qquad a_{k+1} = (k+1)\left(\frac{1}{4}\right)^{k+1}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)\left(\frac{1}{4}\right)^{k+1}}{k\left(\frac{1}{4}\right)^k}\right|$$

$$= \lim_{k\to\infty}\left|\underbrace{\frac{k+1}{k}}_{1}\cdot\left(\frac{1}{4}\right)\right| = \left|\frac{1}{4}\right| = \frac{1}{4} < 1,$$ so this series converges (absolutely)

means the tail of this series ($k\to\infty$) behaves like a geo. series w/ $r=\frac{1}{4}$

ratio test handles factorials well

example $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{(2k+6)!}$ a_k

$$a_{k+1} = \frac{(k+1)!}{(2(k+1)+6)!} = \frac{(k+1)!}{(2k+8)!}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{(k+1)!}{(2k+8)!}}{\frac{k!}{(2k+6)!}}\right| = \lim_{k\to\infty}\left|\frac{(k+1)!}{(2k+8)!}\cdot\frac{(2k+6)!}{k!}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(k+1)!}{k!}\cdot\frac{(2k+6)!}{(2k+8)!}\right|$$

$5! = 5\cdot4\cdot3\cdot2\cdot1$

$k! = k\cdot(k-1)\cdot(k-2)\cdots(1)$

$$= \lim_{k\to\infty}\left|\frac{(k+1)\cancel{(k)}\cancel{(k-1)}\cancel{(k-2)}\cancel{\cdots}\cancel{(1)}}{\cancel{(k)}\cancel{(k-1)}\cancel{(k-2)}\cancel{\cdots}\cancel{(1)}}\cdot\frac{\cancel{(2k+6)}\cancel{(2k+5)}\cancel{\cdots}\cancel{(1)}}{(2k+8)(2k+7)\cancel{(2k+6)}\cancel{(2k+5)}\cancel{\cdots}\cancel{(1)}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{k+1}{(2k+8)(2k+7)}\right| = 0 < 1$$

so series converges

example For what values of x does

$$\sum_{k=1}^{\infty} \frac{5x^k}{4k} \text{ converge?}$$

$$= \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{12}x^3 + \dots$$

ratio test

$$a_k = \frac{5x^k}{4k} \qquad a_{k+1} = \frac{5x^{k+1}}{4(k+1)}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\frac{5x^{k+1}}{4(k+1)}}{\frac{5x^k}{4k}}\right| = \lim_{k\to\infty}\left|\frac{5x^{k+1}}{4k+4}\cdot\frac{4k}{5x^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{4k}{4k+4}\cdot\frac{5x^{k+1}}{5x^k}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1}\cdot x\right| = |x| \quad \text{we want } < 1$$

so, $|x| < 1 \rightarrow -1 < x < 1$

but notice at $x = -1$ and $x = 1$, the ratio is 1

$\rightarrow$ inconclusive

we need to test $x=-1$ and $x=1$ separately

$$\sum_{k=1}^{\infty} \frac{5x^k}{4k}$$

when $x=-1$, $\displaystyle\sum_{k=1}^{\infty} \frac{5}{4} \frac{(-1)^k}{k}$ $\rightarrow$ converges

when $x=1$, $\displaystyle\sum_{k=1}^{\infty} \frac{5}{4k} = \sum_{k=1}^{\infty} \frac{5}{4} \cdot \frac{1}{k}$ $\rightarrow$ diverges (harmonic or p-series w/ $p=1$)

so series converges on $\boxed{-1 \leq x < 1}$

Root Test

given $\sum_{k=1}^{\infty} a_k$, it converges absolutely if ~~$\lim_{k\to\infty} \sqrt[k]{}$~~

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$$

series diverges if $\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$

inconclusive if $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$

why? if $\lim_{k\to\infty} \sqrt[k]{|a_k|} = r$

then for large k, $\sqrt[k]{|a_k|} \approx r$ so $|a_k| \approx r^k$

so like a geo. series w/ ratio r

Root test is good when a_k has some kind of k in exponent

<u>example</u> $\displaystyle\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^{2k}}$

$$a_k = \frac{(k+1)^k}{k^{2k}} = \frac{(k+1)^k}{(k^2)^k} = \left(\frac{k+1}{k^2}\right)^k$$

root test

$$\lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left(\frac{k+1}{k^2}\right)^k} = \lim_{k\to\infty} \frac{k+1}{k^2} = 0 < 1$$

so converges

Ratio Test is good with lots of series, so often it is a good first test to try.

good w/ factorials, things you don't want to integrate, things you don't want to compare to other things

not good w/ things that look like p-series

<u>example</u> $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{\frac{k+1}{(k+1)^2+1}}{\frac{k}{k^2+1}} \right|$$

$$= \lim_{k\to\infty} \left| \underbrace{\frac{k+1}{k}}_{1} \cdot \underbrace{\frac{k^2+1}{(k+1)^2+1}}_{1} \right| = 1 \quad \text{inconclusive}$$

but a comparison tells us the result quickly

$\frac{k}{k^2+1} \approx \frac{k}{k^2} \approx \frac{1}{k}$ as $k \to \infty$, so $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ is like $\sum \frac{1}{k}$ (diverges)